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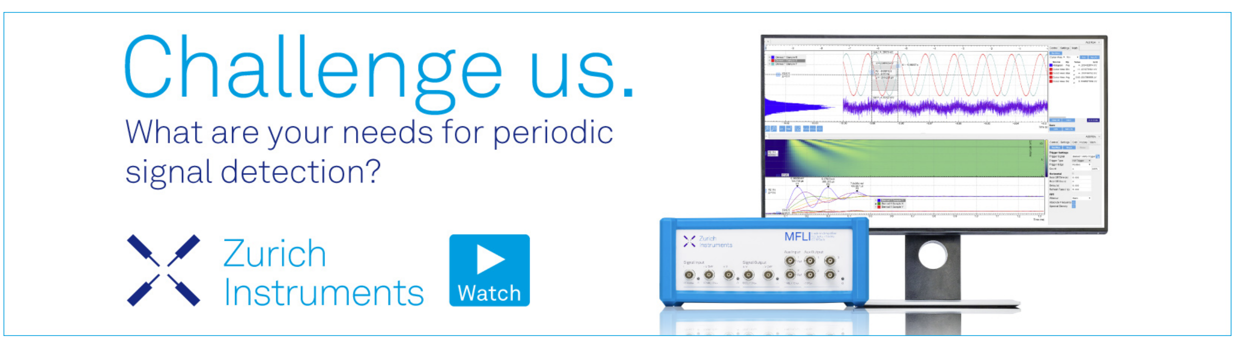
Yu. Bazylevych, and I. Kostiushko



Matrices Diagonalization in Solution of Partial Differential Equation of the First Order

Yu. Bazylevych[1, a)] and I. Kostiushko [2, b)]

[1]Prydniprovska State Academy of Civil Engineering and Architecture, 24A Chernyshevsky str., 49000 Dnipro, Ukraine
[2] Zaporizhzhya National University, 226 Soborny Ave., 69006 Zaporizhzhia, Ukraine

[a)]Corresponding author: bazilvch@ukr.net
[b)] kostushkoia5@gmail.com

Abstract. A new approach to solving systems of linear partial differential equations of the first order has been offered. We use the methods of simultaneous reduction of several matrices. Sometimes, this allows to get an analytical solution or significantly simplify the problem.

INTRODUCTION

Systems of linear partial differential equations are the basis of mathematical models in various application areas [1, 2, 3]. They are usually solved by lowering the order of the equations in the system and using the expansion of the desired functions according to special [4] or trigonometric [5-7] basic functions. One of the most promising schemes, which allows algorithms to calculate and conduct their geometric interpretation, is a reduction to a system of first-order equations [6, 8]. Sometimes, we can get a solution by transformation of matrix pencil.

In this work, we consider the solution of linear partial differential equations of the first order when there are over two matrices and when the order of the matrices is more than two. The case of singular or ill-conditioned matrices is analyzed separately. To simultaneously bring several matrices to a diagonal or block-diagonal form, we use matrix decoupling methods [2, 8, 9].

FORMULATION OF THE PROBLEM

For simplicity, we first consider a system of two partial differential equations for two unknown functions $u = u(x, y)$, $v = v(x, y)$:

$$A\begin{pmatrix} u_x \\ v_x \end{pmatrix} + B\begin{pmatrix} u_y \\ v_y \end{pmatrix} = G\begin{pmatrix} u \\ v \end{pmatrix}. \tag{1}$$

Here A, B, G are constant square matrices.

Nondegenerate linear transformations of the system (1) are:

a) the replacement of variables $\begin{pmatrix} u \\ v \end{pmatrix} = S\begin{pmatrix} U \\ V \end{pmatrix}$, where S is a nonsingular matrix, U, V are new functions of variables x, y,

b) left multiplying of system by a nonsingular matrix H.

In this case, we convert the matrices to the form:

$$\hat{A} = HAS, \quad \hat{B} = HBS, \quad \hat{G} = HGS. \tag{2}$$

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The solution of system (1) is well known in case when G equal to zero and one of others matrices is nonsingular ([3], example 47).

In this article, we consider the solution (or simplification) of equations (1) and similar ones using transformation (2) when there are over two matrices, the order of matrices is over two. We will also consider the case when the original matrices are singular or ill-conditioned.

EXISTING METHODS

Let B_i be complex (generally speaking) n by n matrices. It is necessary either to find such transformation

$$\tilde{B}_i = HB_iS = \mathrm{diag}(B_{i1}, B_{i2}, ..., B_{il}) = \begin{bmatrix} [1] & & 0 \\ & [2] & \\ & & \cdots \\ 0 & & [l] \end{bmatrix}, \quad \max_{S:\,\det(S)\neq 0} l,$$

that all $\{\tilde{B}_i\}$ matrices will have the equal block-diagonal forms, or to prove that such transformation does not exist. We want to get l blocks as maximum as possible number.

We proved that if a B_1 matrix is nonsingular, then for the solution of the problem it is necessary to form auxiliary matrices $C_i = B_1^{-1}B_i \quad i = 2, 3$ and for them to solve similar problems, using only similarity transformations: $\tilde{C}_j = R^{-1}C_jR$.

There exists the method of commuting matrix (see Lopatin [10], Yakubovich [11], Udilov [12] and Bazilevich [2]). The theorems of the theory of matrices are used [13]. Let $\Lambda(C_i)$ be a set of all matrices that are commuting with all matrices $\{C_i\}$. This set we call centralizer. Let a Z matrix be a member of $\Lambda(C_i)$ and have two (or more) different eigenvalues. Vectors of its canonical basis are columns of R transformation matrix of similarity. This method must be used at first to the parent matrices $\{C_i\}$, then to the blocks obtained consistently. We continue this process until receiving only undecoupling blocks. This procedure allows to get the maximal numbers of blocks. A further increase of quantity of blocks is impossible. The uniqueness theorem confirms that.

If all matrices of the original system of differential equations are singular or ill-conditioned, then we use the second theorem of Yakubovich. Firstly, let us introduce the following notation. Let $v(D)$ be the matrix whose columns are the vectors of the canonical basis of the matrix D.

Next, we will need a system of matrix equations

$$T_1B_i = B_iT_2, \quad \left(i = \overline{1, g}\right), \tag{3}$$

where g is the number of source matrices, T_1, T_2 — square matrices of order n .

We can obtain the set of linearly independent solutions of this system by finding the general solution of the corresponding system of linear homogeneous algebraic equations. All elements of the T_1 and T_2 matrices are unknown variables of this system of equations.

Theorem [11]. Let T_1, T_2 be a solution of the equations (3) and $H = v^{-1}(T_1)$, $S = v(T_2)$. If the set of all different eigenvalues of the matrices T_1 and T_2 has more than one element, then the transformation of the matrices B_i

$$\hat{B}_i = HB_iS, \quad \left(i = \overline{1, g}\right) \tag{4}$$

leads them to the following form:

$$\hat{B}_i = \begin{array}{c} \left[\begin{array}{cccc|c} B_{i1} & & & \mathbf{0} & 0 \\ & B_{i2} & & & 0 \\ & & \ddots & & \cdots \\ \mathbf{0} & & & B_{i\,d-1} & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{c} m_1 \\ m_2 \\ \cdots \\ m_{d-1} \\ m_d \end{array} \\ \begin{array}{ccccc} n_1 & n_2 & \cdots & n_{d-1} & n_d \end{array} \end{array}.$$

Here the dimensions $n_k \times m_k$ of the blocks are equal to the multiplicities of the matching eigenvalues of the matrices T_1 and T_2.

The converse is also true: if the transformation (4) reduces the matrices to a block-diagonal form, then there is a nontrivial solution of equations (3).

A CASE WHEN THE NUMBER OF MATRICES IS MORE THAN TWO

Here, we will bring one matrix to the identity matrix. After this, we perform the similarity transformation not for one, but for several matrices. At present, there is no canonical form for simultaneous reduction of several matrices. Therefore, we use the method of commuting matrix.

Example 1. The system of differential equations (1) is considered, in which:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}, \ G = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}.$$

The task is to simplify system (1), by “splitting” it into two independent equations by the method of a commuting matrix and further obtaining its general solution. We will bring the system (1) to the form:

$$A_1 \begin{pmatrix} u_x \\ v_x \end{pmatrix} + E \begin{pmatrix} u_y \\ v_y \end{pmatrix} = G_1 \begin{pmatrix} u \\ v \end{pmatrix}, \tag{5}$$

where $A_1 = B^{-1}A = -\dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\quad G_1 = B^{-1}G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For matrices A_1 and G_1 we will find a centralizer, which means a matrix T, that commutes simultaneously with both matrices:

$$\begin{cases} A_1 T = T A_1; \\ G_1 T = T G_1. \end{cases} \tag{6}$$

We find the general solution of the system (6): $T = aE + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where a and b are free variables. In particular, when $a = 0$, $b = 1$ we get: $T = T^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of the matrix T^* are real and different: $\lambda_{1,2} = \pm 1$, which means the possibility of transforming the original system into two independent equations. From the eigenvectors of the matrix T^*, written in the form of columns, a transformation matrix S is composed: $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

We make the replacement of variables:

$$\begin{pmatrix} u \\ v \end{pmatrix} = S \begin{pmatrix} U \\ V \end{pmatrix}, \tag{7}$$

where U, V are the new functions of variables x, y. We convert the system (5) to the form: $A_2 \begin{pmatrix} U_x \\ V_x \end{pmatrix} + E \begin{pmatrix} U_y \\ V_y \end{pmatrix} = G_2 \begin{pmatrix} U \\ V \end{pmatrix}$, where $A_2 = S^{-1} A_1 S = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $G_2 = S^{-1} G_1 S = -\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. The resulting system contains two independent linear differential equations of the first order: $\begin{cases} -U_x + U_y = -3U; \\ V_y = -V. \end{cases}$ solving of which, taking into account (7), allows analytically obtaining the solution of the source system:

$$u = U + V = e^{3x} f(x+y) + \varphi(x) e^{-y};$$
$$v = U - V = e^{3x} f(x+y) - \varphi(x) e^{-y}.$$

Note that if the method of commuting matrix showed the impossibility of dividing the equations of the system of equations (1) into subsystems, then no replacement of variables makes it possible to divide the system (1) [2].

ORDER OF MATRICES MORE THAN TWO

With a system of differential equations of the third (and more) orders, you can first try using the method of commuting matrix to bring the matrices to a block-diagonal form, divide the equations into two subsystems of the first and second orders. Next, you need to apply the same method for the second-order system (Example 2). Sometimes, a complete decoupling of the system of n differential equations into independent equations is impossible, but the application of this approach allows significant simplification of the original system (Example 3).

Example 2. Consider a system of three equations for unknowns functions of three independent variables x, y, z:

$$E\begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix} + A\begin{pmatrix} u_y \\ v_y \\ w_y \end{pmatrix} + B\begin{pmatrix} u_z \\ v_z \\ w_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8}$$

where $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 8 & 2 & -2 \\ 1 & 8 & -1 \\ 1 & 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 20 & 6 & -6 \\ 3 & 20 & -3 \\ 3 & 6 & 11 \end{pmatrix}$.

For matrices A and B we will find the centralizer T with dimension 3×3: $T^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. Then the transformation S matrix has the form: $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Let us make the replacement of variables:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = S\begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{9}$$

where U, V, W are new functions to be defined. We convert (8) to the form:

$$E\begin{pmatrix} U_x \\ V_x \\ W_x \end{pmatrix} + A_1\begin{pmatrix} U_y \\ V_y \\ W_y \end{pmatrix} + B_1\begin{pmatrix} U_z \\ V_z \\ W_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

where $A_1 = S^{-1}AS = \begin{pmatrix} 8 & 0 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, $B_1 = S^{-1}BS = \begin{pmatrix} 20 & 0 & 0 \\ 6 & 14 & 0 \\ 0 & 0 & 17 \end{pmatrix}$.

System (10) splits into two subsystems:

$$\begin{cases} U_x + 8U_y + 20U_z = 0; \\ V_x + 2U_y + 6V_y + 6U_z + 14V_z = 0; \end{cases} \tag{11}$$

$$\{W_x + 7W_y + 17W_z = 0. \tag{12}$$

We use the method of commuting matrix for the first subsystem (11). The centralizer for matrices $C=\begin{pmatrix}8 & 0\\ 2 & 6\end{pmatrix}$, $F=\begin{pmatrix}20 & 0\\ 6 & 14\end{pmatrix}$ can be represented by the following matrix: $T_1^*=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$. Then the transformation matrix has the form: $S_2=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. We make the replacement of variables

$$\begin{pmatrix}U\\ V\end{pmatrix}=S_2\begin{pmatrix}\tilde U\\ \tilde V\end{pmatrix} \tag{13}$$

where $\tilde U$, $\tilde V$ are new unknown functions. System (11) takes the form:

$$E\begin{pmatrix}\tilde U_x\\ \tilde V_x\end{pmatrix}+\tilde C\begin{pmatrix}\tilde U_y\\ \tilde V_y\end{pmatrix}+\tilde F\begin{pmatrix}\tilde U_z\\ \tilde V_z\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{14}$$

where $\tilde C=S_2^{-1}CS=\begin{pmatrix}8 & 0\\ 0 & 6\end{pmatrix}$, $\tilde F=S_2^{-1}FS=\begin{pmatrix}20 & 0\\ 0 & 14\end{pmatrix}$.

According to (12), (14) the original system (8) has reduced into three independent equations: $\begin{cases}\tilde U_x+8\tilde U_y+20\tilde U_z=0;\\ \tilde V_x+6\tilde V_y+14\tilde V_z=0;\\ W_x+7W_y+17W_z=0,\end{cases}$ solution of which, according to (9), (13), allows to obtain the final solution of the source system (8):

$$\begin{cases}u(x,y,z)=\varphi(y-8x,z-20x)+\psi(y-6x,z-14x);\\ v(x,y,z)=\varphi(y-8x,z-20x)+\vartheta(y-7x,z-17x);\\ w(x,y,z)=\varphi(y-8x,z-20x)+\psi(y-6x,z-14x)+\vartheta(y-7x,z-17x),\end{cases}$$

where φ, ψ, ϑ are arbitrary functions of the specified arguments.

Example 3. The system of equations (8) is considered, in which: $A=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 5\end{pmatrix}$, $B=\begin{pmatrix}3 & 2 & -1\\ 32 & 3 & 4\\ 8 & -2 & 5\end{pmatrix}$.

Having done the calculations similar to the previous one, we get the variable replacement matrix: $S=\begin{pmatrix}1 & 0 & 1\\ -4 & 0 & 4\\ 0 & 1 & 0\end{pmatrix}$.

After that, we have transformed the source system into two subsystems:

$$\begin{cases}U_x-5U_z-V_z=0,\\ V_x+5V_y+16U_z+5V_z=0;\end{cases} \tag{15}$$

$$\{W_x+11W_z=0. \tag{16}$$

The solution of equation (16) has a simple analytical form: $W=F(11x-z;y)$, where F is an arbitrary function. We cannot divide the first second-order subsystem (15) into subsystems. This follows from the fact that any matrix that commutes with the matrixes of coefficients is a multiple of identity matrix. But even in this form, the task is simplified.

THE CASE OF SINGULAR MATRICES

We can formally reduce the problem of obtaining matrices that satisfy (3) to finding a centralizer for auxiliary matrices (see [2], § 6.2):

$$C_1=\begin{pmatrix}E & 0\\ 0 & 2E\end{pmatrix},\ C_2=\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix},\ C_3=\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}. \tag{17}$$

Then the found commuting matrix will look like:

$$W = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}. \tag{18}$$

This is convenient because there are ready-made computer programs for finding the centralizer.

Example 4. The system of differential equations (1) is considered, in which:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},\ G = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Auxiliary matrices defined by formulas (17) are:

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad C_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad C_3 = \begin{pmatrix} 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Commuting matrix W defined according to matrix equations $WC_i = C_iW$ $\left(i = \overline{1,3}\right)$.

The basis of the C_1, C_2, C_3 matrices centralizer is the following: $W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $W_2 = E$.

According to (18), we find: $T_1 = T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The transformation matrix S, composed of eigenvectors of the matrix T_i $\left(i = 1,2\right)$ and written in the form of columns, has the form: $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. In the initial system (1) the following variables were replaced:

$$\begin{pmatrix} u \\ \upsilon \end{pmatrix} = S \begin{pmatrix} f \\ g \end{pmatrix}, \tag{19}$$

where f and g are new functions to be defined. Then we perform the left multiplication of the transformed system by the matrix $H = S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We come to the solution of the new system: $\hat{A}\begin{pmatrix} f_x \\ g_x \end{pmatrix} + \hat{B}\begin{pmatrix} f_y \\ g_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\hat{A} = HAS = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{B} = HBS = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. The original system of differential equations is transformed to two independent equations: $\begin{cases} 2f_x = 0; \\ 4g_y = 0, \end{cases}$ which have obvious solutions: $f = \varphi(y)$, $g = \psi(x)$, where φ, ψ are arbitrary functions According to the substitution (19), which determines the general solution of the initial system of equations: $u = \varphi(y) + \psi(x)$; $\upsilon = \varphi(y) - \psi(x)$.

CONCLUSIONS

We consider a new approach to solving systems of linear first-order partial differential equations. Using matrix methods, we divide the original system of equations into subsystems, which simplifies the process of their further solutions. To continue this work we plan to use the method of transforming matrices to block-triangular form [14].

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